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Oscillation theorems for second order neutral differential equations

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ABSTRACT

The aim of this paper is to study the oscillation of the second order neutral differential equations

$$\left(r(t)\,[x(t)+p(t)x(\tau(t))]'\right)' + q(t)x(\sigma(t)) = 0. \tag{E}$$

The obtained results are based on the new comparison theorems, that enable us to reduce the problem of the oscillation of the second order equation to the oscillation of the first order equation. The obtained comparison principles essentially simplify the examination of the studied equations.

1. Introduction

In this paper, we shall study the oscillation behavior of the solutions of the second order neutral differential equations of the form

$$\left(r(t)\,[x(t)+p(t)x(\tau(t))]'\right)' + q(t)x(\sigma(t)) = 0, \tag{E}$$

where $q(t) \in C([t_0,\infty))$, $r(t), p(t), \tau(t), \sigma(t) \in C^1([t_0,\infty))$ and

(H_1) $r(t) > 0$, $q(t) > 0$, $0 \le p(t) \le p_0 < \infty$;
(H_2) $\lim_{t\to\infty} \tau(t) = \infty$, $\lim_{t\to\infty} \sigma(t) = \infty$,
(H_3) $\tau'(t) \ge \tau_0 > 0$, $\tau \circ \sigma = \sigma \circ \tau$.

For our further references we denote and assume that

$$R(t) = \int_{t_0}^{t} \frac{1}{r(t)} \mathrm{d}s \to \infty \quad \text{as } t \to \infty. \tag{1.1}$$

We set $z(t) = x(t) + p(t)x(\tau(t))$. By a solution of Eq. (E) we mean a function $x(t) \in C([T_x,\infty))$, $T_x \ge t_0$, which has the property $r(t)z'(t) \in C^1([T_x,\infty))$ and satisfies (E) on $[T_x,\infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_x$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$ and otherwise, it is said to be nonoscillatory. Eq. (E) itself is said to be oscillatory if all of its solutions are oscillatory.

The second order equations have the applications in various problems of physics, biology, economy. Therefore, there is constant interest in obtaining sufficient conditions for the oscillation or nonoscillation of the solutions of various types of the second order equations. See e.g. papers [1–19].

Known oscillation criteria require various restrictions on the coefficients of the studied neutral differential equations.

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Grammatikopoulos et al. [1] have shown that $0 \le p(t) \le 1$ together with $\int^{\infty} q(s)\big(1 - p(s-\sigma)\big)\mathrm{d}s = \infty$ guarantee the oscillation of the neutral equation

$$\big(x(t) + p(t)x(t-\tau)\big)'' + q(t)x(t-\sigma) = 0.$$

For the same equation, Erbe et al. [2] established the oscillation criterion that requires

$$q(t) \ge q_0 > 0, \qquad p_1 \le p(t) \le p_2, \quad p(t) \text{ not eventually negative.}$$

This result has been improved and generalized by other authors. We mention Grace and Lalli [3] who studied oscillation of

$$\big(r(t)\,[x(t) + p(t)x(t-\tau)]'\big)' + q(t)f(x(t-\sigma)) = 0,$$

under the conditions

$$\frac{f(x)}{x} \ge k, \qquad \int^{\infty} \frac{\mathrm{d}s}{r(s)} = \infty,$$

and

$$\int^{\infty} \rho(s)q(s)\big(1 - p(s-\sigma)\big) - \frac{(\rho'(s))^2 r(s-\sigma)}{4k\rho(s)}\mathrm{d}s = \infty,$$

where $\rho(t)$ is an optional function.

Xu and Xia [4] established oscillation of

$$\big(x(t) + p(t)x(t-\tau)\big)'' + q(t)f(x(t-\sigma)) = 0,$$

provided that

$$0 \le p(t) < \infty, \qquad q(t) \ge M > 0.$$

Li et al. [5] studied the more general neutral differential equation

$$\big(r(t)\,[x(t) + p(t)x(\tau(t))]'\big)' + q(t)f(x(\sigma(t))) = 0. \tag{E$_1$}$$

Using double Riccati transformation, they presented new oscillation criteria, where they replaced usually used restriction $0 \le p(t) \le 1$ with $0 \le p(t) \le p_0 < \infty$ and also required inter alia $\sigma(t) \le \tau(t) \le t$ and

$$\int^{\infty} \rho(s)q(s)\big(1 - p(s-\sigma)\big) - \frac{(\rho'(s))^2 r(s-\sigma)}{4k\rho(s)}\mathrm{d}s = \infty,$$

where $\rho(t)$ is an optional function.

In this paper we use a different technique for studying the oscillation of (E). We shall establish new comparison theorems in which we compare the second order equation (E) with the first order differential inequality in the sense that the absence of the positive solutions of this first order inequality yields the oscillation of (E). Established comparison theorems essentially simplify the examination of (E) and enable us also to eliminate some conditions imposed in the cited papers on the coefficients of (E). Moreover, our results can be easily extend to cover also the more general differential equations of the form (E$_1$).

Remark 1. All functional inequalities considered in this paper are assumed to hold eventually, that is they are satisfied for all t large enough.

Remark 2. In the proofs of our on-coming results, when eliminating all nonoscillatory solutions of (E), we can deal only with positive ones.

2. Main results

It follows from (1.1) that the positive solutions of (E) have the following property.

Lemma 1. *If $x(t)$ is a positive solution of (E), then the corresponding function $z(t) = x(t) + p(t)x(\tau(t))$ satisfies*

$$z(t) > 0, \qquad r(t)z'(t) > 0, \qquad \big(r(t)z'(t)\big)' < 0, \tag{2.1}$$

eventually.

Proof. Assume that $x(t)$ is a positive solution of (E). Then it follows from (E) that

$$\big(r(t)z'(t)\big)' = -q(t)x(\sigma(t)) < 0.$$

Consequently, $r(t)z'(t)$ is decreasing and thus either $z'(t) > 0$ or $z'(t) < 0$ eventually. If we let $z'(t) < 0$, then also $r(t)z'(t) < -c < 0$ and integrating this from t_1 to t, we have

$$z(t) \le z(t_1) - c\int_{t_1}^{t} \frac{1}{r(s)}\mathrm{d}s \to -\infty \quad \text{as } t \to \infty.$$

This contradicts the positivity of $z(t)$ and the proof is complete. □

For our further references, let us denote

$$Q(t) = \min\{q(t), q(\tau(t))\}, \tag{2.2}$$

where t is large enough.

Theorem 1. *Let $Q(t)$ be defined in (2.2) and t_1 be large enough. Assume that the first order neutral differential inequality*

$$\left(y(t) + \frac{p_0}{\tau_0}y(\tau(t))\right)' + Q(t)\big(R(\sigma(t)) - R(t_1)\big)y(\sigma(t)) \le 0, \tag{E$_2$}$$

has no positive solution, then (E) is oscillatory.

Proof. Assume that $x(t)$ is a positive solution of (E). Then the corresponding function $z(t)$ satisfies

$$\begin{aligned} z(\sigma(t)) &= x(\sigma(t)) + p(\sigma(t))x(\tau(\sigma(t))) \\ &\le x(\sigma(t)) + p_0x(\sigma(\tau(t))), \end{aligned} \tag{2.3}$$

where we have used the hypothesis (H_3).

On the other hand, it follows from (E) that

$$\big(r(t)z'(t)\big)' + q(t)x(\sigma(t)) = 0, \tag{2.4}$$

and moreover taking (H_1) and (H_3) into account, we have

$$\begin{aligned} 0 &= \frac{p_0}{\tau'(t)}\big(r(\tau(t))z'(\tau(t))\big)' + p_0q(\tau(t))x(\sigma(\tau(t))) \\ &\ge \frac{p_0}{\tau_0}\big(r(\tau(t))z'(\tau(t))\big)' + p_0q(\tau(t))x(\sigma(\tau(t))). \end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), we are led to

$$\big(r(t)z'(t)\big)' + \frac{p_0}{\tau_0}\big(r(\tau(t))z'(\tau(t))\big)' + q(t)x(\sigma(t)) + p_0q(\tau(t))x(\sigma(\tau(t))) \le 0,$$

which in view of (2.3) and (2.2) provides

$$\big(r(t)z'(t)\big)' + \frac{p_0}{\tau_0}\big(r(\tau(t))z'(\tau(t))\big)' + Q(t)z(\sigma(t)) \le 0. \tag{2.6}$$

It follows from Lemma 1 that $y(t) = r(t)z'(t) > 0$ is decreasing and then

$$\begin{aligned} z(t) &\ge \int_{t_1}^{t} \frac{1}{r(s)}\big(r(s)z'(s)\big)\,\mathrm{d}s \ge y(t)\int_{t_1}^{t}\frac{1}{r(s)}\mathrm{d}s \\ &= y(t)\big(R(t) - R(t_1)\big). \end{aligned} \tag{2.7}$$

Therefore, (2.6) together with (2.7) ensures that $y(t)$ is a positive solution of (E_2). This contradicts our assumptions and the proof is complete. □

Remark 3. The condition $\tau \circ \sigma = \sigma \circ \tau$ of the hypothesis (H_3) is satisfied, e.g. for $\tau(t) = at$ along with $\sigma(t) = bt$.

Remark 4. When studying properties of the neutral differential equations there are usually further restrictions on the coefficients of (E) like $\sigma(t) \le \tau(t)$, $\sigma(t) \le t$, $\tau(t) \le t$, $0 \le p(t) < 1$, etc. included. In Theorem 1 no such constraints is involved and what is more we do not stipulate whether $\tau(t)$ is delay or advanced argument and accordingly $\sigma(t)$ can be delay argument or $\sigma(t) - t$ can oscillate. Consequently, our result is of high generality and extends and complements the known ones. Moreover, the comparison principle established in Theorem 1 reduces the oscillation of (E) to the research of the first order neutral differential inequality (E_2). Therefore, applying the existing or new conditions for (E_2) to have no positive solution, we immediately get oscillation criteria for (E).

Applying additional conditions on the coefficients of (E), we can deduce from Theorem 1 various oscillation criteria for (E).

Theorem 2. *Let $Q(t)$ be defined in (2.2) and t_1 be large enough. Further assume that*

$$\tau(t) \ge t. \tag{2.8}$$

If the first order differential inequality

$$w'(t) + \frac{\tau_0}{\tau_0 + p_0} Q(t)\big(R(\sigma(t)) - R(t_1)\big) w(\sigma(t)) \le 0 \tag{E_3}$$

has no positive solution, then (E) is oscillatory.

Proof. We assume that $x(t)$ is a positive solution of (E). It follows from Lemma 1 and the proof of Theorem 1 that $y(t) = r(t)z'(t) > 0$ is decreasing and it satisfies (E_2). Let us denote $w(t) = y(t) + \frac{p_0}{\tau_0} y(\tau(t))$. It follows from (2.8) that

$$w(t) \le y(t)\left(1 + \frac{p_0}{\tau_0}\right).$$

Substituting these terms into (E_2), we get that $w(t)$ is a positive solution of (E_3). A contradiction. □

Adding restriction that $\sigma(t)$ is a delay argument, and using suitable criterion for the absence of positive solutions of (E_3) (see e.g. [6] or [7]), we get easily verifiable oscillation result for (E).

Corollary 1. *Assume that $Q(t)$ is defined in (2.2) and t_1 is large enough. Further assume that (2.8) holds and*

$$\sigma(t) \le t. \tag{2.9}$$

If

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} Q(s)R(\sigma(s))\mathrm{d}s > \frac{\tau_0 + p_0}{\tau_0\, \mathrm{e}}, \tag{2.10}$$

then (E) is oscillatory.

Proof. It is easy to see that if (2.10) holds, then also

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} \frac{\tau_0}{\tau_0 + p_0} Q(s)\big(R(\sigma(s)) - R(t_1)\big)\mathrm{d}s > \frac{1}{\mathrm{e}}.$$

But this condition according to Theorem 2.1.1 from [7] guarantees that (E_3) has no positive solution and the assertion now follows from Theorem 2. □

Now, we turn our attention to the case when $\tau(t)$ is delay argument. We use the notation $\tau^{-1}(t)$ for its inverse function.

Theorem 3. *Let $Q(t)$ be defined in (2.2) and t_1 be large enough. Further assume that*

$$\tau(t) \le t. \tag{2.11}$$

If the first order differential inequality

$$w'(t) + \frac{\tau_0}{\tau_0 + p_0} Q(t)\big(R(\sigma(t)) - R(t_1)\big) w(\tau^{-1}\sigma(t)) \le 0, \tag{E_4}$$

has no positive solution, then (E) is oscillatory.

Proof. We assume that $x(t)$ is a positive solution of (E). Then $y(t) = r(t)z'(t) > 0$ is a decreasing solution of (E_2). We denote $w(t) = y(t) + \frac{p_0}{\tau_0} y(\tau(t))$. What is more (2.11) implies

$$w(t) \le y(\tau(t))\left(1 + \frac{p_0}{\tau_0}\right).$$

Substituting this into (E_2), we get that $w(t)$ is a positive solution of (E_3). A contradiction. □

Corollary 2. *Assume that $Q(t)$ is defined in (2.2) and t_1 is large enough. Further assume that*

$$\sigma(t) \le \tau(t) \le t. \tag{2.12}$$

If

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^{t} Q(s)R(\sigma(s))\mathrm{d}s > \frac{\tau_0 + p_0}{\tau_0\, \mathrm{e}}, \tag{2.13}$$

then (E) is oscillatory.

The proof of the corollary is very similar to the proof of Corollary 1 and so it can be omitted.

Example 1. We consider the second order neutral differential equation

$$\left(t^{1/2}\left[x(t)+p_0x(\alpha t)\right]'\right)'+\frac{a}{t^{3/2}}x(\beta t)=0, \tag{E$_5$}$$

where $0<p_0<\infty$, $0<\alpha<\infty$, $0<\beta<1$, and $a>0$.

If $\alpha\geq 1$, then $Q(t)=q(\tau(t))=a/(\alpha t)^{3/2}$ and moreover, condition (2.10) of Corollary 1 reduces to

$$\frac{2a\beta^{1/2}}{\alpha^{3/2}}\ln\frac{1}{\beta}>\frac{\alpha+p_0}{\alpha\,\mathrm{e}},$$

which guarantees the oscillation of (E$_5$).

On the other hand, if $0<\beta<\alpha\leq 1$, then $Q(t)=q(t)=a/t^{3/2}$ and condition (2.13) of Corollary 2 reduces to

$$2a\beta^{1/2}\ln\frac{\alpha}{\beta}>\frac{\alpha+p_0}{\alpha\,\mathrm{e}},$$

which also guarantees the oscillation of (E$_5$). Consequently, we have covered the oscillation of (E$_5$) for all $\alpha\in(0,\infty)$ that is for $\tau(t)=\alpha t$ to be delay or advanced argument. Note that the oscillation criterion from [5] requires condition (2.12) and so it cannot be applied when $\alpha\geq 1$. On the other hand, the criterion from [1] needs $0\leq p(t)\leq 1$ and therefore, it fails when $p_0>1$. Moreover, the oscillation condition from [4] wants $q(t)\geq M>0$ and so it does not work for (E$_5$) and from the same reason oscillation condition from [2] fails.

Theorem 1 permits to obtain new oscillation criteria for (E) provided that (E$_2$) has no positive solutions. In Theorems 2 and 3 we have established our own new criteria for the absence of positive solutions of (E$_2$). But we can also apply such existing criteria for (E$_2$) to achieve a new oscillation result for (E). We illustrate this fact on the following two results.

Theorem 4. *Assume that*

$$1<p_1\leq p(t)\leq p_2<\infty, \tag{2.14}$$

$$\sigma(t)<\tau(t)<t, \tag{2.15}$$

$$\int^{\infty} q(s)\mathrm{d}s=\infty, \tag{2.16}$$

then (E) is oscillatory.

Proof. Theorem 3.6.1 in [2] implies that conditions (2.14)–(2.16) together with (H_2) guaranty that (E$_2$) has no positive solution. Our assertion follows now from Theorem 1. □

Theorem 5. *Assume that (2.16) holds and*

$$0\leq p(t)\leq p_0<1, \tag{2.17}$$

$$\tau(t)>t,\qquad \sigma(t)<t, \tag{2.18}$$

then (E) is oscillatory.

Proof. It follows from Theorem 2.4.3 in [8] that conditions (2.16)–(2.18) together with (H_2) ensure that (E$_2$) has no positive solution. The rest of the proof follows now from Theorem 1. □

Remark 5. All our conclusions can be very easily extended to nonlinear neutral differential equation (E$_1$), namely

$$\left(r(t)\left[x(t)+p(t)x(\tau(t))\right]'\right)'+q(t)f(x(\sigma(t)))=0.$$

Adding additional condition

$$\frac{f(x)}{x}\geq k,\quad x\neq 0,$$

the reader can verify that our results here hold also for (E$_1$), provided that we replace in the assumptions of our achievements the function $q(t)$ by $kq(t)$.

3. Summary

In this paper we have introduced new comparison theorems for investigation of the oscillation of (E). The established comparison principles reduce the oscillation of the second order neutral equations to studying the properties of various types of the first order differential inequalities, which essentially simplifies the examination of (E). Our technique permits us to relax restrictions usually imposed on the coefficients of (E). So that our results are of high generality and can be easily extended also to the nonlinear neutral differential equations. The obtained results are easily applicable and are illustrated on a suitable example.

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